

# Control of Squeezed States

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## Abstract

In this paper we consider the classical and quantum control of squeezed states of harmonic oscillators. This provides a method for reducing noise below the quantum limit and provides an example of the control of under-actuated systems in the stochastic and quantum context. We consider also the interaction of a squeezed quantum oscillator with an external heat bath.

## 1 Introduction

In this paper we consider the problem of squeezing of harmonic oscillators from the point of view of control theory. Squeezing has been suggested as a method for reducing noise in quantum systems below the standard quantum limit. This can be achieved by using laser pulses and in that sense may be viewed as a quantum control problem, although the classical squeezing problem is also of interest. In the latter case one is interested in reducing noise induced by random perturbations.

The quantum control problem has been of great interest recently, see for example Brockett and Khaneja [1999], Lloyd [1996] and Warren et. al. [1993] and references therein.

Here we consider squeezing as a control problem in both the classical and quantum setting. In the classical case we consider a system subject

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to thermal noise while in the quantum case we consider a system at zero temperature and in the presence of noise. In both cases the control is given by an external electromagnetic field and enters the control equations multiplicatively. In this sense the setting is similar to the NMR control problems analyzed by Brockett and Khaneja.

A key feature of squeezing is that it results in a redistribution of uncertainty between observables.

In this paper we consider a model for phonon squeezing in solids following the work of Garret et al. [1997], but one can equally well consider the case of photons in quantum optics. The control is via a single pulse on a large ensemble of oscillators and this sense we are considering under-actuated control systems in both the classical and quantum case.

We also model the effect of dissipation on the classical system and the effect of coupling to a heat bath in the quantum setting. This causes the squeezing effect to gradually moderate.

## 2 The Control Setting

In the classical setting we consider bilinear control systems of the form

$$\dot{x}^i = \sum_{j=1}^n a_{ij}x^j + \sum_{j=1}^m b_{ij}(x)u^j + \sum_{j=1}^r g_{ij}\dot{w}^j \quad (2.1)$$

where  $a_{ij}$  is a constant matrix (i.e. the free dynamics is linear),  $b_{ij}$  is linear in  $x$ , i.e. the control  $u^j$  enters bilinearly,  $\dot{w}^k$  is white noise and the state space is  $\mathbb{R}^n$ .

In the quantum context we want to consider a similar equation but defined on an appropriate Hilbert space:

$$i\hbar \frac{\partial \psi}{\partial t} = H_s \psi + \sum_j u_j H_j \psi + \sum_j \dot{w}^j H_j \psi \quad (2.2)$$

where  $H_s$  is the Schroedinger operator, the  $H_j$  are (linear) input operators,  $u^j$  are functions of time, and the  $\psi$  is a vector in the Hilbert space.

## 3 Classical Squeezing of the Harmonic Oscillator

In this section we consider classical squeezing of a set of identical coupled harmonic oscillators. Denote the position of each oscillator by  $u^i$ .

The Hamiltonian for the system is of the form:

$$H = \sum_i \frac{p_i^2}{2} + \sum_i \frac{\omega_i^2}{2} (q^i - q^{i+1})^2 \quad (3.1)$$

where the oscillators are assumed to have unit mass and  $p_i = \dot{u}^i$ .

In order to analyze the system we decompose it into its normal modes. Denoting the normal mode coordinates by  $Q^i$  we thus obtain a system of uncoupled harmonic oscillator equations of the form  $\ddot{Q}^i + \Omega_i^2 Q^i = 0$ .

The main control mechanism we consider here is squeezing by pulses. In this case each oscillator is forced by a pulse at time  $t = 0$  which is proportional to its displacement, i.e. we have equations of the form:

$$\ddot{Q}^i + \Omega_i^2 Q^i = 2\lambda Q^i \delta(t) \quad (3.2)$$

where  $\delta(t)$  is the Dirac delta function and  $\lambda$  is a constant which is proportional to the frequency  $\Omega$ .

Thus we obtain

$$\dot{Q}^i(0^+) = \dot{Q}^i(0^-) + 2\lambda Q^i(0). \quad (3.3)$$

Thus, if one considers the system subject to white noise,

$$\ddot{Q}^i + \Omega_i^2 Q^i = 2\lambda Q^i \delta(t) + \alpha \dot{w}^i, \quad (3.4)$$

one sees that while one starts with a spherical equilibrium distribution which is invariant in time, after the pulse one has an elliptical distribution which rotates in time at twice the harmonic frequency (by the  $\mathbb{Z}_2$  symmetry of the ellipse). (A precise analysis is given below in the course of our treatment of the quantum mechanical case.) Noise reduction is then achieved by viewing the system “stroboscopically” when the noise is low.

Actually the above is an idealization: in actuality the oscillator should be viewed as in equilibrium with a heat bath which dissipates energy. In the classical setting one can model this by simple linear dissipation (in the quantum setting one has to introduce a heat bath – see below).

Thus we have a system of the form

$$\ddot{Q}^i + \Omega_i^2 Q^i = -\eta_i \dot{Q}^i + U_i(t) + \alpha \dot{w}^i \quad (3.5)$$

where  $\eta_i$  is a dissipation constant and  $U_i(t)$  is the control which we can choose to be a single pulse or a sequence of pulses. Depending on the dissipation

strength an initial squeezing effect will decay away and we need a continual sequences of pulse to keep the system in a squeezed state.

It is worthwhile remarking on the how the control enters in our setting: the control is a single pulse applied overall (and in this sense the system is under-actuated) while the effect on each (normal mode) of oscillation is to apply a pulse proportional to displacement (minus the mean displacement which is of course zero for each oscillator). This is effected by the type of interaction of the oscillators with the field that the pulse induces. We note also that in the full nonlinear setting the mean displacement may not be zero and must be taken into account.

We shall return to the classical squeezing of oscillator by pulses, and in particular a computation of mean square displacement, after a discussion of the quantum case below.

We note also that parametric resonance control can achieve similar squeezing effects in the classical case. In this case we consider oscillator motion in the presence of a modulating drive:

$$\ddot{Q} - \omega^2(1 + \epsilon \cos 2\omega t)Q = 0 \quad (3.6)$$

where  $\epsilon$  parameterizes the strength of the drive. We omit details of this approach here.

## 4 Squeezing of the Quantum Harmonic Oscillator

We now turn to the quantum setting.

Consider the following Hamiltonian

$$H = \frac{P^2}{2m} + \frac{m\omega^2}{2}Q^2 + \lambda\delta(t)Q^2, \quad (4.1)$$

which reflects an impulsive change in the spring constant and where  $\omega = \sqrt{K/m}$ ,  $K$  being the original spring constant.

The variables  $P$  and  $Q$ , which are operators in the quantum case, obey canonical commutation rules  $[P, Q] = i\hbar$ . We can rewrite the above Hamiltonian in terms of creation operators  $a$  and  $a^\dagger$  defined through

$$Q = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger), \quad P = i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger - a), \quad (4.2)$$

with  $[a, a^\dagger] = 1$ . Written in terms of the new variables, the Hamiltonian is

$$H = \hbar\omega(a^\dagger a + 1/2) + \lambda\delta(t)(a + a^\dagger)^2. \quad (4.3)$$

The ground state of the system, for  $t \neq 0$ ,  $|0\rangle$ , corresponds to the vacuum of  $a$ ,  $(a|0\rangle = 0)$ , and the excited states are of the form  $(a^\dagger)^2|0\rangle$ .

We now want to study the behavior of the system at  $t > 0$ , given that the system is in its ground state at  $t < 0$ . The wave function at  $t = 0^+$  is of the form  $|\psi(t = 0^+)\rangle = \exp(-i\lambda Q^2)|0\rangle$ , and for longer times the system evolves with the “unperturbed” Hamiltonian:  $|\psi(t > 0)\rangle = \exp(-iH_0 t)e^{-i\lambda Q^2}|0\rangle$ . Our first quantity of interest is  $\langle\psi(t)|Q^2|\psi(t)\rangle \equiv \langle Q^2(t)\rangle$ . Let us compute it using the general method of coherent states. We find

$$\langle Q^2(t)\rangle = \langle 0|e^{i\lambda Q^2}(ae^{-i\omega t} + a^\dagger e^{-i\omega t})^2 e^{-i\lambda Q^2}|0\rangle, \quad (4.4)$$

where we have used the fact that  $e^{iH_0 t}ae^{-iH_0 t} = ae^{-i\omega t}$ , which states that  $a^\dagger$  and  $a$  respectively destroy and create eigenstates of  $H_0$ , and where  $Q$  is defined in units of  $\sqrt{\hbar/(2m\omega)}$ .

Now we introduce a basis of coherent states  $|z\rangle$ , which satisfy  $a|z\rangle = z|z\rangle$ ,  $\langle z|a^\dagger = \langle z|z^*$ , and form an overcomplete set of states:

$$1 = \frac{1}{2\pi i} \int dz dz^* e^{-zz^*} |z\rangle\langle z|. \quad (4.5)$$

Inserting (4.5) in (4.4) we find

$$\begin{aligned} \langle Q^2(t)\rangle &= \frac{1}{2\pi i} \int \int dz dz^* e^{-zz^*} \\ &\quad (z^2 e^{-2i\omega t} + z^{*2} e^{2i\omega t} + 2zz^* - 1) |\langle 0|e^{i\lambda x^2}|z\rangle|^2. \end{aligned}$$

In order to evaluate the last term we need the position representation of the ground state (note that at this point  $Q$  is a real number)

$$\langle 0|Q\rangle = \frac{1}{\pi^{1/4}} e^{-Q^2/2} \quad (4.6)$$

and that of the coherent state

$$\langle Q|z\rangle = \frac{1}{\pi^{1/4}} e^{-Q^2/2 + \sqrt{2}zQ - z^2/2}. \quad (4.7)$$

A simple integration gives

$$\langle 0|e^{i\lambda Q^2}|z\rangle = \int dx \langle 0|Q\rangle \langle Q|z\rangle e^{i\lambda Q^2} \quad (4.8)$$

$$= \frac{1}{\sqrt{1-i\lambda}} e^{i\lambda z^2/2(1-i\lambda)}. \quad (4.9)$$

Changing to the variables  $z = u + iv$  we have

$$e^{-zz^*} |\langle 0 | e^{i\lambda Q^2} | z \rangle|^2 = \frac{1}{\sqrt{1+\lambda^2}} e^{-[v^2 + (2\lambda^2+1)u^2 + 2\lambda uv]/(1+\lambda^2)}, \quad (4.10)$$

and

$$\begin{aligned} \langle Q^2(t) \rangle &= \frac{4}{\pi\sqrt{1+\lambda^2}} \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \\ &\quad \left( u^2 \cos^2 \omega t + v^2 \sin^2 \omega t + uv \sin 2\omega t - \frac{1}{4} \right) \\ &\quad \times e^{-[v^2 + (2\lambda^2+1)u^2 + 2\lambda uv]/(1+\lambda^2)} \end{aligned} \quad (4.11)$$

$$= 1 + 4\lambda^2 \sin^2 \omega t + 2\lambda \sin 2\omega t \quad (4.12)$$

It is interesting to compare this with an ensemble of classical oscillators with initial conditions taken from a heat bath. For simplicity let us take  $\omega = m = k_B = T = 1$  ( $k_B$  is Boltzman's constant). An arbitrary oscillator will evolve as  $Q(t) = u \cos t + v \sin t$ , with  $u$  and  $v$  its initial position and velocity. If a pulse is applied at  $t = 0$  of the form treated above:  $Q(t) = u \cos t + (v + 2\lambda u) \sin t$ . Now let us average over initial conditions taken from a measure given by (a thermal bath)

$$\begin{aligned} \langle Q^2(t) \rangle &\sim \int du dv [u \cos t + (v + 2\lambda u) \sin t]^2 e^{-(u^2+v^2)} \\ &= 1 + 4\lambda^2 \sin^2 t + 2\lambda \sin 2t. \end{aligned} \quad (4.13)$$

It is interesting to note that the two expressions for, respectively, the quantum oscillator at zero temperature and the classical oscillator at finite temperature, are exactly the same. The general time dependence of the variance for a squeezed harmonic oscillator with frequency  $\omega$  can thus be written in the following form

$$\langle [Q(t)]^2 \rangle = \frac{\epsilon_0}{K} \left[ 1 + \left( \frac{2\lambda}{\omega} \right) \sin 2\omega t + \left( \frac{2\lambda}{\omega} \right)^2 \sin^2 \omega t \right] \quad (4.14)$$

with  $\epsilon_0 = \hbar\omega/2$  for the quantum case and  $\epsilon_0 = k_B T$  for the classical oscillator at a temperature  $T$ .

The method of coherent states presented above has the advantage of being suitable for calculating other quantities. For example, if the oscillators are atoms within a solid, the scattering amplitude for an X-ray is decreased by a factor (called the Debye-Waller factor – see Ziman [1972])

$\sim \langle \exp ikQ(t) \rangle$ , with  $k$  the wave-vector of the X-ray. We now ask ourselves what is the time evolution of the Debye-Waller factor for a squeezed phonon. This means that we need to compute the following expression

$$\begin{aligned}
I(\lambda, t) &= \langle 0 | e^{i\lambda Q^2} e^{(ae^{-i\omega t} + a^\dagger e^{-i\omega t})} e^{-i\lambda Q^2} | 0 \rangle \\
&= \frac{1}{\sqrt{e}} \frac{1}{\sqrt{1+\lambda^2}} \frac{1}{\pi} \int du dv \\
&\quad e^{2u \cos \omega t + 2v \sin \omega t - \frac{[v^2 + (2\lambda^2 + 1)u^2 + 2\lambda uv]}{(1+\lambda^2)}} \\
&= e^{1+4\lambda^2 \sin^2 \omega t + 2\lambda \sin 2\omega t}.
\end{aligned} \tag{4.15}$$

For the Debye-Waller factor, we obtain the following time dependence

$$\langle e^{ikQ(t)} \rangle = e^{-k^2 \langle Q^2(t) \rangle} \tag{4.16}$$

Measurement of the Debye-Waller factor may provide a practical method of detecting the squeezing phenomenon experimentally.

## 5 Squeezing and dissipation

In this section we consider the squeezing of a quantum oscillator coupled to a an infinite number of oscillators representing a “heat” bath. We show that this causes a decay in the squeezing oscillation for small time and true damping in the limit of a continuum of oscillators. This damping effect of the heat bath is similar to that analyzed classically in Lamb [1900], Komech [1995], Sofer and Weinstein [1999] and Hagerty, Bloch and Weinstein [1999]. We stress that we are considering a zero temperature case, and the damping effects appear due to a) the coupling of a single variable with a continuum of variables and b) an “asymmetry” in the initial conditions. The applied pulse on the oscillator generates outgoing waves on the continuum system which in turn gives rise to a positive damping (for a detailed discussion of negative versus positive damping see Keller and Bonilla [1986]).

We start with a general formulation, and at the end of this section discuss a specific continuum example.

The Hamiltonian of the system consists of three parts:  $H_0$  describing the original oscillator:

$$H_0 = \frac{p_0^2}{2m} + \frac{m\omega_0^2}{2} q_0^2, \tag{5.1}$$

the Hamiltonian  $H_e$  of the environment:

$$H_e = \sum_{\alpha} \left[ \frac{p_{\alpha}^2}{2m} + \frac{m\omega_{\alpha}^2}{2} q_{\alpha}^2 \right], \quad (5.2)$$

and a linear coupling between the two

$$H_{\text{int}} = \sum_{\alpha} \xi_{\alpha} q_{\alpha} q_0. \quad (5.3)$$

Formally, the total Hamiltonian  $H = H_0 + H_e + H_{\text{int}}$  can be written in terms of its normal mode coordinates  $X_{\nu}$  and  $P_{\nu}$ :

$$H = \sum_{\nu} \left[ \frac{P_{\nu}^2}{2m} + \frac{m\omega_{\nu}^2}{2} X_{\nu}^2 \right], \quad (5.4)$$

and we will consider a situation in which the initial (before the pulse) wave function corresponds to all the modes in the ground state:

$$\Psi_0 = \prod_{\nu} \left( \frac{\omega_{\nu}}{\pi\hbar} \right)^{1/4} e^{-\omega_{\nu} X_{\nu}^2 / 2\hbar}. \quad (5.5)$$

At  $t = 0$  a pulse is applied to the (original) oscillator, the wave function immediately after the pulse given by:

$$\Psi_0(t = 0^+) = e^{i\lambda q_0^2} \Psi_0 \quad (5.6)$$

$$= e^{i\lambda \sum_{\mu\nu} U_{0\mu} U_{0\nu} X_{\mu} X_{\nu}} \Psi_0, \quad (5.7)$$

where  $U_{\mu\nu}$  is the matrix transforming from the original (uncoupled) modes to the coupled system ( $q_0 = \sum_{\nu} U_{0\nu} X_{\nu}$ ).

As in previous sections, we are interested in the fluctuations of the variance of  $q_0$ , given in this case by

$$\langle q_0^2(t) \rangle = \sum_{\mu\nu} U_{0\mu} U_{0\nu} \langle X_{\mu} X_{\nu} \rangle(t), \quad (5.8)$$

and that we will compute by solving the equation of motion obeyed by the correlations  $\langle X_{\mu} X_{\nu} \rangle(t)$ . Since  $X_{\mu}$  and  $X_{\nu}$  correspond to harmonic coordinates, using the quantum mechanical commutation relations we compute



the equations of motion to be:

$$\begin{aligned}
\frac{d}{dt}\langle X_\mu X_\nu \rangle &= \frac{1}{m}\langle (P_\mu X_\nu + P_\nu X_\mu) \rangle \\
\frac{d^2}{dt^2}\langle X_\mu X_\nu \rangle &= -(\omega_\mu^2 + \omega_\nu^2)\langle X_\mu X_\nu \rangle + \frac{2}{m^2}\langle P_\mu P_\nu \rangle \\
\frac{d}{dt}\langle P_\mu P_\nu \rangle &= -m(\omega_\mu^2\langle X_\mu P_\nu \rangle + \omega_\nu^2\langle X_\nu P_\mu \rangle) \\
\frac{d^2}{dt^2}\langle P_\mu P_\nu \rangle &= -(\omega_\mu^2 + \omega_\nu^2)\langle P_\mu P_\nu \rangle + 2m^2\omega_\mu^2\omega_\nu^2\langle X_\mu X_\nu \rangle.
\end{aligned}$$

Note that the above equations are identical to those of classical harmonic oscillators for the quantities  $X_\mu(t)X_\nu(t)$  etc., with initial conditions given by the values of the correlations evaluated for the quantum wave function:

$$\begin{aligned}
\langle X_\mu X_\nu \rangle(0^+) &= \delta_{\mu\nu} \frac{\hbar}{2m\omega_\mu}, \\
\langle P_\mu P_\nu \rangle(0^+) &= \delta_{\mu\nu} \frac{\hbar m\omega_\mu}{2} \\
&+ 2\hbar^2\lambda^2(1 + \delta_{\mu\nu}) \frac{U_{0\mu}}{m\omega_\mu} \frac{U_{0\nu}}{m\omega_\nu} q_0^2 \\
\langle (X_\mu P_\nu + P_\nu X_\mu) \rangle(0^+) &= 4\lambda\hbar U_{0\mu}U_{0\nu} \frac{\hbar}{2m} \left( \frac{1}{\omega_\mu} + \frac{1}{\omega_\nu} \right)
\end{aligned}$$

with  $q_0^2 \equiv \langle q_0^2(0^-) \rangle = \sum_\alpha \hbar U_{0\alpha}^2 / 2m\omega_\alpha$ .

Collecting the above equations we obtain

$$\langle q_0^2(t) \rangle = q_0^2 \left\{ 1 + 4\lambda^2 S^2(t) + \frac{\lambda}{q_0^2} C(t) S(t) \right\}, \quad (5.9)$$

with

$$S(t) = \sum_\mu \frac{\hbar U_{0\mu}^2}{m\omega_\mu} \sin \omega_\mu t \quad C(t) = \sum_\mu \frac{\hbar U_{0\mu}^2}{m\omega_\mu} \cos \omega_\mu t.$$

All the information of the evolution of the variance is contained in the function  $J(\omega)$ , the physical interpretation of which is that of a local density of states of the oscillator, defined as

$$J(\omega) = \sum_\mu \frac{\hbar U_{0\mu}^2}{m\omega_\mu} \delta(\omega - \omega_\mu), \quad (5.10)$$

from which

$$S(t) = \int d\omega J(\omega) \sin \omega t, \quad C(t) = \int d\omega J(\omega) \cos \omega t. \quad (5.11)$$

Note that  $J(\omega)$  is a sum over delta functions, giving rise to a superposition of oscillations with the frequencies  $\omega_\nu$  for both  $S(t)$  and  $C(t)$ . In the limit of an infinite system, and when the modes are spatially extended over all space  $J(\omega)$  becomes a continuous function. In that case the oscillatory behavior acquires a damped component, the detailed time dependence being given by the frequency spectrum of  $J(\omega)$ . A lorentzian shape for  $J(\omega)$  will give an exponentially damped oscillation for both  $S(t)$  and  $C(t)$ . As an illustration of this point we consider a model for which  $J(\omega)$  can be computed explicitly – see the classical analysis in Lamb [1900] Komech [1995]. Consider a one-dimensional string coupled to our oscillator. The string is described by a “transverse” displacement  $u(x, t)$ . The classical equations of motion of the system are

$$\begin{aligned} u_{tt}(x, t) &= c^2 u_{xx}(x, t) \\ M d^2 q_0(t)/dt^2 &= -V q_0(t) + T[u_x(0+, t) - u_x(0-, t)] \\ q_0(t) &= u(0, t). \end{aligned} \quad (5.12)$$

The normal modes consist of even and odd (in  $x$ ) solutions. The odd solutions do not involve  $q_0$  and are of the form  $u_{q,o}(x, t) = e^{icqt} \sin qx$ , whereas the even solutions are of the form  $u_{q,e}(x, t) = e^{icqt} \cos(q|x| + \delta_q)$ , with  $\delta_q$  a phase shift (to be found). The wave vectors  $q$  label the normal modes, and play the role of the index  $\mu$  in the above discussion:  $\omega_\mu = cq$ , and  $U_{\mu 0}^2 = \cos^2(\delta_q)$  (up to a normalization constant) in the present case. Substituting this expression in (5.12) we obtain ( $\omega_0^2 = V/M$ )

$$\tan \delta_q = \frac{Mc}{2T} \frac{(\omega_0^2 - \omega_q^2)}{\omega_q}, \quad (5.13)$$

from which  $U_{\mu 0}^2 = \cos^2 \delta_q$  is given by

$$U_{\mu 0}^2 = \frac{\alpha^2 \omega_q^2}{\alpha^2 \omega_q^2 + (\omega_q^2 - \omega_0^2)^2} \equiv U_q^2, \quad (5.14)$$

where we have defined  $\alpha = 2T/Mc$ . Note that  $U_q$  represents the transformation matrix that has to be normalized and since the frequencies form a

continuum we normalize  $U_q(\omega_q)$  to its integral over  $\omega_q$ . Omitting the index  $q$  in  $\omega_q$ , we obtain

$$U(\omega) = \frac{2\alpha}{\pi} \frac{\omega^2}{\alpha^2\omega^2 + (\omega^2 - \omega_0^2)^2} = \frac{m\omega}{\hbar} J(\omega). \quad (5.15)$$

Substituting (5.15) in (5.11) we obtain

$$S(t) = \frac{\hbar}{m\omega_0} e^{-\Gamma t} \sin \Omega_0 t, \quad C(t) = \frac{\hbar}{m\omega_0} e^{-\Gamma t} \cos \Omega_0 t, \quad (5.16)$$

with

$$\Omega_0 = \omega_0 \left(1 + [\alpha/\omega_0]^2\right)^{1/4} \cos \delta/2, \quad (5.17)$$

$$\Gamma = \omega_0 \left(1 + [\alpha/\omega_0]^2\right)^{1/4} \sin \delta/2, \quad (5.18)$$

where  $\delta = \tan^{-1} \alpha/\omega_0$ .

In the realistic limit  $\alpha \ll \omega_0$  which corresponds to a “weak” coupling to the environment) this expressions take the form:  $S(t) \cong (\hbar/(m\omega_0) \exp(-Tt/Mc) \sin \omega_0 t, C(t) \cong (\hbar/(m\omega_0) \exp(-Tt/Mc) \cos \omega_0 t$ .

Note that in this model, and in the limit of weak coupling, the initial variance  $q_0^2$  of the reference oscillator is unchanged due to the coupling to the environment, and is given by  $q_0^2 = \hbar/2m\omega_0$ . Our final result for this section is then

$$\langle q_0^2(t) \rangle \cong q_0^2 \left\{ 1 + e^{-2(T/Mc)t} \left[ \left( \frac{2\lambda\hbar}{m\omega_0} \right) \sin 2\omega_0 t + \left( \frac{2\lambda\hbar}{m\omega_0} \right)^2 \sin^2 \omega_0 t \right] \right\}, \quad (5.19)$$

which reduces simply to (4.13) in the uncoupled case of  $T = 0$ .

In summary we have shown in this section that the coupling to the environment can be included in general, giving rise to dissipation, and that the squeezing effect in the presence of dissipation can be computed explicitly for the Lamb model.

Additional details of the analysis here, extensions to the squeezing of a nonlinear oscillator, and a treatment of the quantum measurement issue will appear in forthcoming publications.

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